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On Spectral Radius of Graphs

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Abstract—Let G(V, E) be simple graph with n vertices and m edges and A be vetex subset of V(G). For any v∈ A the degree of the vertex v_i with respect to the subset A is defined as the number of vertices A that are adjacent to v_i . We call it as $\mathfrak{D}$ –degree and is denoted by $\mathfrak{D}_i$. Denote $\lambda_1(G)$ as the largest eigenvalue of the graph G and s_i.as the sum of $\mathfrak{D}$ degree of vertices that are adjacent to v_i. In this paper we give lower bounds of $\lambda_1(G)$ in terms of $\mathfrak{D}$ degree.

Index Terms—$\mathfrak{D}$-degree, spectral radius, dominating set, covering set, adjacency matrix.

1 Introduction

Let G be simple graph with n vertices and m edges. For any $v_i \in V$, the degree of v_i, denoted by d_i, is the number of edges that are adjacent to v_i . The new definition of the degree of a vertex and results using this can be seen in papers[8, 9]. A subset D of V is called dominating set if every vertex of V-D is adjacent to to some vertex in D. Any dominating set with minimum cardinality is called minimum dominating set and this cardinality is called domination number. A subset C of V is called covering set if every edge of V is incident to at least one vertex of C. Any covering set with minimum cardinality is called minimum covering set and the number of element of this set is called covering number.

For any graph G, let A(G)=(a_{ij}) be the adjacency matrix. The eigenvalues λ_1, $\lambda_2, \ldots, \lambda_n$ of A assumed in non-increasing order. The largest eigenvalue $\lambda_1(G)$ is called spectral radius of the graph. Till now, there are many bounds for $\lambda_1(G)$ (see for instance [3,6]). Further results and properties see the reviews [1,2,4,5]. In this paper, we give bounds for λ_1 of G in tems of $\mathfrak{D}$-degree, which is generalization of earlier bounds found in the paper [3,6]. Using minimum dominating set and minimum covering set, the adjacency matrices $A_D(G)$, $A_C(G)$ have been defined in the papers [7,10]. The $\mathfrak{D}$-degree is also used to get bounds for the largest eigenvalue $\lambda_1^D(G)$, $\lambda_1^C(G)$ of these matrices.

We start with defining degree of vertex with respect to a subset of vertex set or $\mathfrak{D}$-degree.

Definition1.1: Let G(V, E) be a simple graph and A be any vertex sub-set. The degree of a vertex v_i of a graph G with respect to A is the number of vertices of A that are adjacent to v_i. This degree is denoted by $d_A(v_i)$ *or* $\mathfrak{D}_i$.

Here A may or may not be the subset of V. However, if A=∅ then $\mathfrak{D}_i$ =0 and if A=V then $\mathfrak{D}_i$ = d(v_i) ∀ $v_i \in V$.

2 Bounds for Spectral Radius

Theorem 2.1. Let G be a connected graph with n vertices as $v_1, v_2, \ldots, v_n$. If $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_n$ represents $\mathfrak{D}$-degree sequence of these vertices with respect to a vertex subset A of G then

$$\lambda_1(G) \geq \sqrt{\frac{s_1^2 + s_2^2 + \ldots + s_n^2}{\mathfrak{D}_1^2 + \mathfrak{D}_2^2 + \ldots + \mathfrak{D}_n^2}}$$

Where, s_i is sum of $\mathfrak{D}$ degree vertices that are adjacent to v_i. Equality case occurs when A=V

Proof. Let $X = (x_1, x_2, \ldots, x_n)^T$ be a unit positive eigenvector of A(G) corresponding to λ_1(G).

Let $C = \sqrt{\frac{1}{\sum_{i=1}^n \mathfrak{D}_i^2}} (\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_n)^T$

$$AC = \sqrt{\frac{1}{\sum_{i=1}^n \mathfrak{D}_i^2}} \left(\sum_{j=1}^n a_{1j}\mathfrak{D}_j, \ldots, \sum_{j=1}^n a_{nj}\mathfrak{D}_j\right)^T$$

$$= \sqrt{\frac{1}{\sum_{i=1}^n \mathfrak{D}_i^2}} (s_1, s_2, \ldots, s_n)^T$$

Similarly $C^T A = \sqrt{\frac{1}{\sum_{i=1}^n \mathfrak{D}_i^2}} (s_1, s_2, \ldots, s_n)$

$$C^T A^2 C = \frac{1}{\sum_1^n \mathfrak{D}_i^2} (s_1^2 + s_2^2 + \cdots + s_n^2)$$

$$\therefore \lambda_1(G) = \sqrt{\lambda_1(A^2)} = \sqrt{X^T A^2 X} \geq \sqrt{C^T A^2 C}$$

Hence $\lambda_1(G) \geq \sqrt{\frac{s_1^2 + s_2^2 + \ldots + s_n^2}{\mathfrak{D}_1^2 + \mathfrak{D}_2^2 + \ldots + \mathfrak{D}_n^2}}$

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Corollary 2.1. When A= V, $\lambda_1(G) \geq \sqrt{\frac{t_1^2+t_2^2+\ldots+t_n^2}{\mathfrak{D}_1^2+\mathfrak{D}_2^2+\ldots+\mathfrak{D}_n^2}}$

where t_i represents the sum of degree of vertices that are adjacent to v_i.

Theorem 2.2. Let G be a connected graph with $\mathfrak{D}$-degree sequence of vertices as $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \ldots, \mathfrak{D}_n$ corresponding to the subset A of G then

$$\lambda_1(G) \geq \sqrt{\frac{(d_1\mathfrak{D}_1 + d_2\mathfrak{D}_2 + \cdots + d_n\mathfrak{D}_n)^2}{n(\mathfrak{D}_1^2 + \mathfrak{D}_2^2 + \ldots + \mathfrak{D}_n^2)}}$$

Proof. By theorem2.1, $\lambda_1(G) \geq \sqrt{\frac{s_1^2+s_2^2+\ldots+s_n^2}{\mathfrak{D}_1^2+\mathfrak{D}_2^2+\ldots+\mathfrak{D}_n^2}}$

By Cauchy's Schwarz inequality, $\sum_{i=1}^n s_i^2 \geq \frac{(\sum_i^n s_i)^2}{n}$

$\therefore \lambda_1(G) \geq \sqrt{\frac{(s_1+s_2+\cdots+s_n)^2}{n(\mathfrak{D}_1^2+\mathfrak{D}_2^2+\ldots+\mathfrak{D}_n^2)}}$

Also $s_1 + s_2 + \cdots + s_n = d_1\mathfrak{D}_1 + d_2\mathfrak{D}_2 + \cdots + d_n\mathfrak{D}_n$

$$\lambda_1(G) \geq \sqrt{\frac{(d_1\mathfrak{D}_1 + d_2\mathfrak{D}_2 + \cdots + d_n\mathfrak{D}_n)^2}{n(\mathfrak{D}_1^2 + \mathfrak{D}_2^2 + \ldots + \mathfrak{D}_n^2)}}$$

Corollary 2.2. For a connected graph G and a vertex subset A with |A|= k then $\lambda_1(G) \geq \frac{\sum_{i=1}^k d_i}{n}$

Proof: By theorem 2.2

$$\lambda_1(G) \geq \sqrt{\frac{(d_1\mathfrak{D}_1 + d_2\mathfrak{D}_2 + \cdots + d_n\mathfrak{D}_n)^2}{n(\mathfrak{D}_1^2 + \mathfrak{D}_2^2 + \ldots + \mathfrak{D}_n^2)}}$$

Since $d_i \geq \mathfrak{D}_i$, $\lambda_1(G) \geq \sqrt{\frac{(\mathfrak{D}_1^2+\mathfrak{D}_2^2+\cdots+\mathfrak{D}_n^2)^2}{n(\mathfrak{D}_1^2+\mathfrak{D}_2^2+\ldots+\mathfrak{D}_n^2)}}$

Which implies $\lambda_1(G) \geq \sqrt{\frac{(\mathfrak{D}_1^2+\mathfrak{D}_2^2+\cdots+\mathfrak{D}_n^2)}{n}}$

By Cauchy's Scwarz inequality, $\sum_{i=1}^n \mathfrak{D}_i^2 \geq \frac{(\sum_i^n \mathfrak{D}_i)^2}{n}$

Thus $\lambda_1(G) \geq \sqrt{\frac{(\mathfrak{D}_1+\mathfrak{D}_2+\cdots+\mathfrak{D}_n)^2}{n^2}}$

$$\lambda_1(G) \geq \frac{\sum_{i=1}^n \mathfrak{D}_i}{n}$$

But $\sum_{i=1}^n \mathfrak{D}_i = \sum_{i=1}^k d_i$ where k = |A| $\lambda_1(G) \geq \frac{\sum_{i=1}^k d_i}{n}$

Corollary 2.3. If A = D , a minimum dominating set,

Then $\sum_{i=1}^k d_i \geq (n-1)$ and so $\lambda_1(G) \geq \frac{(n-1)}{n}$.

Corollary 2.4. If A=V then $\sum_{i=1}^n \mathfrak{D}_i = \sum_{i=1}^n d_i = 2m$ therefore, $\lambda_1(G) \geq \frac{2m}{n}$.

We recall the definition of minimum dominating matrix $A_D(G)$, as defined in [7].

Definition 2.1. The minimum dominating adjacency matrix $A_D(G)$is defined by $A_D(G) = D(G)$+ A(G) , where A(G) is the adjacency matrix and D(G) =(d_{ij}) is $n \times n$ matrix with

$$d_{ij} = \begin{cases} 1 & if \;\; i = j \;\; and \;\; v_i \in D \\ 0 & otherwise \end{cases}$$

Theorem 2. 3. Let G be a connected graph with n vertices and D be a minimum dominating set. If $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_n$ represents $\mathfrak{D}$-degree sequence of these vertices corresponding to the vertex subset D of G and $A_D(G)$ is minimum dominating matrix then

$$\lambda_1^D(G) \geq \sqrt{\frac{\sum_{i=1}^n s_i^2 + \sum_{i=1}^k \mathfrak{D}_i^2}{\sum_{i=1}^n \mathfrak{D}_i^2}} \text{ where k = |D|}$$

Where, s_i is sum of $\mathfrak{D}$ degree vertices that are adjacent to v_i.
Equality case occurs when D =V.

Proof. Let X = $(x_1, x_2, \ldots, x_n)^T$ be a unit positive eigenvector of $A_D(G)$ corresponding to $\lambda_1^D(G)$).

Let C = $\sqrt{\frac{1}{\sum_{i=1}^n \mathfrak{D}_i^2}}(\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_n)^T$

$$(D+A)C = \sqrt{\frac{1}{\sum_{i=1}^n \mathfrak{D}_i^2}}\left(\sum_{i=1}^n (d_{1j} + a_{1j})\mathfrak{D}_j, \ldots, \sum_{j=1}^n (d_{nj} + a_{nj})\mathfrak{D}_j\right)^T$$

$$= \sqrt{\frac{1}{\sum_{i=1}^n \mathfrak{D}_i^2}}\left(s_1 + \sum_{j=1}^n d_{1j}\mathfrak{D}_j, \ldots, s_n + \sum_{j=1}^n d_{nj}\mathfrak{D}_j\right)^T$$

Similarly ,

$$C^T(D+A) = \sqrt{\frac{1}{\sum_{i=1}^n \mathfrak{D}_i^2}}\left(s_1 + \sum_{j=1}^n d_{ij}\mathfrak{D}_j, \ldots, s_n + \sum_{j=1}^n d_{nj}\mathfrak{D}_j\right)$$

Therefore $C^T(D+A)^2 C$ =

$$\frac{1}{\sum_1^n \mathfrak{D}_i^2}\left[\left(s_1 + \sum_{j=1}^n d_{ij}\mathfrak{D}_j\right)^2 + \cdots + \left(s_n + \sum_{j=1}^n d_{nj}\mathfrak{D}_j\right)^2\right]$$

$$\geq \frac{1}{\sum_1^n \mathfrak{D}_i^2}\left[s_1^2 + s_2^2 \ldots + s_n^2 + \left(\sum_{j=1}^n d_{1j}\mathfrak{D}_j\right)^2 + \cdots + \left(\sum_{j=1}^n d_{nj}\mathfrak{D}_j\right)^2\right]$$

Hence $C^T(D+A)^2 C \geq \frac{1}{\sum_1^n \mathfrak{D}_i^2}\left[s_1^2 + s_2^2 \ldots + s_n^2 + \sum_{i=1}^k \mathfrak{D}_i^2\right]$

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where k = |D|

$$\therefore\ \lambda_1^D(G) = \sqrt{\lambda_1(D+A)^2} = \sqrt{X^T(D+A)^2X}$$

$$\geq\ \sqrt{C^T(D+A)^2C}$$

$$\therefore\ \lambda_1^D(G) \geq \sqrt{\frac{\sum_{i=1}^n s_i^2 + \sum_{i=1}^k \mathfrak{D}_i^2}{\sum_{i=1}^n \mathfrak{D}_i^2}}$$

Theorem 2. 4. Let G be a connected graph with n vertices and D be minimum dominating set. If $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_n$ represents $\mathfrak{D}$-degree sequence of these vertices corresponding to the vertex subset D of G and $A_D(G)$is minimum dominating matrix then

$$\lambda_1^D(G) \geq \frac{(d_1\mathfrak{D}_1 + d_2\mathfrak{D}_2 + \cdots + d_n\mathfrak{D}_n + \sum_{i=1}^k \mathfrak{D}_i^2)}{\sqrt{n\sum_{i=1}^n \mathfrak{D}_i^2}} \text{ where } k = |D|$$

Proof. By theorem 2.3 $\lambda_1^D(G) \geq \sqrt{\frac{\sum_{i=1}^n s_i^2 + \sum_{i=1}^k \mathfrak{D}_i^2}{\sum_{i=1}^n \mathfrak{D}_i^2}}$

By Cauchy Schwarz inequality, the above relation simplifies to

$$\lambda_1^D(G) \geq \sqrt{\frac{(\sum_{i=1}^n s_i^2 + \sum_{i=1}^k \mathfrak{D}_i^2)^2}{n\sum_{i=1}^n \mathfrak{D}_i^2}}$$

Also $s_1 + s_2 + \cdots + s_n = d_1\mathfrak{D}_1 + d_2\mathfrak{D}_2 + \cdots + d_n\mathfrak{D}_n$

$$\lambda_1^D(G) \geq \frac{(d_1\mathfrak{D}_1 + d_2\mathfrak{D}_2 + \cdots + d_n\mathfrak{D}_n + \sum_{i=1}^k \mathfrak{D}_i^2)}{\sqrt{n\sum_{i=1}^n \mathfrak{D}_i^2}}$$

Theorem 2.5. Let G be a connected graph and D represents minimum dominating set with k = |D| then

$$\lambda_1^D(G) \geq \frac{2m+k}{n}$$

Proof. Since $d_i \geq \mathfrak{D}_i$, we get new bounds for $\lambda_1^D(G)$

$$\lambda_1^D(G) \geq \frac{(\sum_{i=1}^n \mathfrak{D}_i^2 + \sum_{i=1}^k \mathfrak{D}_i^2)}{\sqrt{n\sum_{i=1}^n \mathfrak{D}_i^2}}$$

But for any real numbers a and b$\frac{a+b}{\sqrt{a}} \geq \sqrt{(a+b)}$ so

$$\frac{(\sum_{i=1}^n \mathfrak{D}_i^2 + \sum_{i=1}^k \mathfrak{D}_i^2)}{\sqrt{\sum_{i=1}^n \mathfrak{D}_i^2}} \geq \sqrt{(\sum_{i=1}^n \mathfrak{D}_i^2 + \sum_{i=1}^k \mathfrak{D}_i^2)}$$

$$\lambda_1^D(G) \geq \frac{\sqrt{(\sum_{i=1}^n \mathfrak{D}_i^2 + \sum_{i=1}^k \mathfrak{D}_i^2)}}{\sqrt{n}}$$

Using Cauchy Schawarz inequality

$$\left(\sum_{i=1}^n \mathfrak{D}_i^2 + \sum_{i=1}^k \mathfrak{D}_i^2\right) \geq \frac{(\sum_{i=1}^n \mathfrak{D}_i + \sum_{i=1}^k \mathfrak{D}_i)^2}{n}$$

The above relations simplifies to $\lambda_1^D(G) \geq \frac{(\sum_{i=1}^n \mathfrak{D}_i + \sum_{i=1}^k \mathfrak{D}_i)}{n}$

Since $\sum_{i=1}^n \mathfrak{D}_i = \sum_{i=1}^k d_i$where k = |D|

$$\lambda_1^D(G) \geq \frac{\sum_{i=1}^k (d_i + \mathfrak{D}_i)}{n}$$

When D = V, $\sum_{i=1}^n \mathfrak{D}_i = \sum_{i=1}^n d_i$ =2m and

$$\sum_{i=1}^k \mathfrak{D}_i = \sum_{i=1}^k d_i \geq k$$

Therefore $\lambda_1^D(G) \geq \frac{2m+k}{n}$

The above results is true for minimum covering matrix [10]. Hence we have the following theorem.

Theorem2.6. Let G be a connected graph and C be a minimum covering set. The minimum adjacency covering matrix, $A_C(G)$ defined by $A_C(G) = D(G)$+ A(G) , where A(G) is the adjacency matrix and D(G) =(d_{ij}) is $n \times n$ matrix with

$$d_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{otherwise} \end{cases}$$

Then $\lambda_1^C(G) \geq \sqrt{\frac{\sum_{i=1}^n s_i^2 + \sum_{i=1}^c \mathfrak{D}_i^2}{\sum_{i=1}^n \mathfrak{D}_i^2}}$

$$\lambda_1^C(G) \geq \frac{(d_1\mathfrak{D}_1 + d_2\mathfrak{D}_2 + \cdots + d_n\mathfrak{D}_n + \sum_{i=1}^c \mathfrak{D}_i^2)}{\sqrt{n\sum_{i=1}^n \mathfrak{D}_i^2}} \text{ where } c = |C|$$

$$\lambda_1^D(G) \geq \frac{2m+c}{n}$$

where, c is covering number

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4 CONCLUSION

In this paper we obtained lower bounds for largest eigen value(Spectral radius) in terms of $\mathfrak{D}$-degree. It was proved that these bounds are generalization of earlier bounds found in the papers[5,6].

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